

# A refined energy bound for distinct perpendicular bisectors

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## Abstract

Let  $\mathcal{P}$  be a set of  $n$  points in the Euclidean plane. We prove that either a single line or circle contains  $(1 - \delta)n$  points of  $\mathcal{P}$ , or the number of distinct perpendicular bisectors determined by pairs of points in  $\mathcal{P}$  is  $\Omega(n^{52/35-\epsilon})$ , for any  $\delta, \epsilon > 0$ . This is the first substantial progress toward the proof a conjecture of Lund, Sheffer, and de Zeeuw [11], that either a single line or circle contains all but an arbitrarily small, constant fraction of the points of  $\mathcal{P}$ , or  $\mathcal{P}$  determines  $\Omega(n^2)$  distinct bisectors. In proving this result, we introduce a new method of applying an energy bound to a refined subset of the pairs of points of  $\mathcal{P}$ , identified using an incidence bound.

We also suggest a new approach to the Erdős pinned distance problem.

## 1 Introduction

Many classic problems in discrete geometry ask for the minimum number of distinct equivalence classes of subsets of a fixed set of points under some geometrically defined equivalence relation. The seminal example is the Erdős distinct distance problem [4]: How few distinct distances can be determined by a set of  $n$  points in the Euclidean plane? Guth and Katz have nearly resolved the Erdős distinct distance question [6], but there are numerous other examples of questions of this type, many of which remain wide open.

One natural question that has not recieved much attention is: How few distinct perpendicular bisectors can be determined by a set of  $n$  points in the Euclidean plane?

Without any additional assumption, it is not too hard to give a complete answer to this question. The vertices of a regular  $n$ -gon determine  $n$  distinct perpendicular bisectors. Each point of an arbitrary point set  $\mathcal{P}$  determines  $n - 1$  distinct bisectors with the remaining points of  $\mathcal{P}$ , and this is tight when  $n = 2$ . In subsection 1.2, we give a simple geometric argument showing that the number of distinct bisectors is at least  $n$ , when  $n > 2$ .

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Suppose we assume that no circle or line contains more than  $K$  points of  $\mathcal{P}$ . In this case, the author, Sheffer, and de Zeeuw give the following lower bound on  $|\mathcal{B}|$ , the number of distinct bisectors determined by  $\mathcal{P}$ , a fixed set of  $n$  points in the Euclidean plane [11]<sup>1</sup>:

$$|\mathcal{B}| = \Omega \left( \min \left\{ K^{-\frac{2}{5}} n^{\frac{8}{5}-\varepsilon}, K^{-1} n^2 \right\} \right). \quad (1)$$

We further offered the following

**Conjecture 1.** *For any  $\varepsilon > 0$ , there is a constant  $c_\varepsilon > 0$  such that either a single line or circle contains  $(1 - \varepsilon)n$  points of  $\mathcal{P}$ , or  $|\mathcal{B}| \geq c_\varepsilon n^2$ .*

In this paper, we take a significant qualitative step toward Conjecture 1 by proving

**Theorem 2.** *For any  $\delta, \varepsilon > 0$ , either a single circle or line contains  $(1 - \delta)n$  points of  $\mathcal{P}$ , or*

$$|\mathcal{B}| = \Omega(n^{52/35-\varepsilon}),$$

where the constants hidden in the  $\Omega$ -notation depend on  $\delta, \varepsilon$ .

The proof (in [11]) of inequality (1) uses the, now standard, method of bounding the “energy”<sup>2</sup> of the quantity in question. In particular, we write  $\mathcal{B}(a, b)$  for the perpendicular bisector of distinct points  $a, b$ , and define the *bisector energy* to be the size of the set

$$\mathcal{Q} = \{(a, b, c, d) \in \mathcal{P}^4 : a \neq b, c \neq d, \mathcal{B}(a, b) = \mathcal{B}(c, d)\}.$$

It is easy to see that  $|\mathcal{Q}| \leq n^2(n - 1)$ , since each element of  $\mathcal{Q}$  is determined by  $(a, b, c)$ ; taking  $\mathcal{P}$  to be the vertices of a regular  $n$ -gon shows that this bound is tight. In [11], we show

$$|\mathcal{Q}| \leq O \left( K^{\frac{2}{5}} n^{\frac{12}{5}+\varepsilon} + K n^2 \right), \quad (2)$$

and conjecture that the strongest possible bound is  $|\mathcal{Q}| \leq O(K n^2)$ . A standard application of Cauchy-Swartz (see, for example, the proof of Lemma 9, below) gives

$$|\mathcal{B}| \geq n^2(n - 1)^2 / |\mathcal{Q}|.$$

Using this inequality, it is a straightforward calculation to obtain (1) from (2).

<sup>1</sup>In a somewhat different direction, Hanson, the author, and Roche-Newton gave a non-trivial lower bound for the number of (algebraically defined) perpendicular bisectors determined by a large subset of a finite plane [7].

<sup>2</sup>The term *additive energy*, referring to the number of quadruples  $(a, b, c, d)$  in some underlying set of numbers such that  $a + b = c + d$ , was coined by Tao and Vu [8]. Starting with the work of Sharir, Elekes [3] and Guth, Katz [6] on the distinct distance problem, the strategy of using geometric incidence bounds to obtain upper bounds on analogously defined energies has become indispensable in the study of questions about the number of distinct equivalent subsets.

Observe that even a tight bound of  $|\mathcal{Q}| \leq O(Kn^2)$  would only give  $|\mathcal{B}| \geq \Omega(n^2K^{-1})$ . This is qualitatively very different than the bound proposed in Conjecture 1, which is  $\Omega(n^2)$  unless  $K \geq (1 - \varepsilon)n$ . Hence, it initially seems hopeless to use an energy bound to prove (or even make substantial progress toward) Conjecture 1.

The main new idea in this paper is to apply an energy bound to a refined subset of the pairs of points of  $\mathcal{P}$ . We show that there is a large set  $\Pi \subset \mathcal{P} \times \mathcal{P}$  of pairs of points, such that

$$\mathcal{Q}^* = \{(a, b, c, d) \in \mathcal{P}^4 : (a, b), (c, d) \in \Pi, \mathcal{B}(a, b) = \mathcal{B}(c, d)\}$$

is small. In particular, we define  $\Pi$  to be the set of pairs of points of  $\mathcal{P}$  that are not contained in any circle or line that contains too many points of  $\mathcal{P}$ . We use a point-circle incidence bound, proved in [1], to show that  $\Pi$  must be large, and use an argument similar to that bounding  $\mathcal{Q}$  in [11] to show that  $\mathcal{Q}^*$  must be small.

The proof of Theorem 2 is in Section 2.

## 1.1 Application to pinned distances

In Section 3, we give an application of the methods and results of this paper to a problem of Erdős on the set of distances determined by a set of points in the plane.

Let  $P$  be a set of  $n$  points in the Euclidean plane. We denote

$$\begin{aligned} \delta(p) &= \{\|x - p\| : x \in P\} \\ \delta^* &= \max_{p \in P} |\delta(p)| \end{aligned}$$

Erdős conjectured that  $\delta^* = \Omega(n/\log(n))$  for all point sets. The best current result on this problem is by Katz and Tardos [10], who built on the work of Solymosi and Tóth [13]; Katz and Tardos showed that  $\delta^* = \Omega(n^{0.864})$ .

We show

**Theorem 3.** *For any  $\varepsilon > 0$ ,*

$$\delta^* = O(n^{87/105 - \varepsilon}).$$

Although this is weaker than the result of Katz and Tardos, the proof is different, and there is a clear path to improvement. In particular, both Theorem 2 and Theorem 3 depend on a point-circle incidence bound (Theorem 7) and a refined bisector energy bound (Lemma 10). Neither Theorem 7 nor Lemma 10 is tight, and if optimal versions of these results were proved, we would immediately get nearly optimal results for pinned distances and distinct perpendicular bisectors. A more detailed discussion of these issues is given in Section 4.

In the proof of Theorem 3, we use a new weighted Szemerédi-Trotter bound. This gives an upper bound on the number of incidences between weighted points and lines, when we have a bound on the sum of squares of the weights. Researchers working on similar problems in incidence geometry may find this a convenient tool. The statement and proof of the result are in Section 5.

## 1.2 There are at least $n$ bisectors

We give the best possible general lower bound on  $|\mathcal{B}|$ .

**Proposition 4.** *If  $n > 2$ , then  $|\mathcal{B}| \geq n$ .*

*Proof.* Since any point  $a \in \mathcal{P}$  determines  $n - 1$  distinct bisectors with the remaining points  $\mathcal{P} \setminus \{a\}$ , it is sufficient to show that there are three points  $a, b, c$  such that  $\mathcal{B}(b, c)$  is distinct from  $\mathcal{B}(a, x)$  for any  $x \in \mathcal{P}$ . If there are three collinear points, this is immediate, so we assume that no three points are collinear.

Let  $a, b \in \mathcal{P}$  so that  $|ab|$  is minimal, and let  $c \in \mathcal{P}$  so that the angle  $\angle abc$  is minimal. If  $a$  is on the same side of  $\mathcal{B}(b, c)$  as  $c$ , then  $|ac| \leq |ab|$ , which is a contradiction. If  $a$  is on the line  $\mathcal{B}(b, c)$ , then there is no point  $x$  such that  $\mathcal{B}(a, x) = \mathcal{B}(b, c)$ , and we have accomplished our goal. Hence, we may suppose that  $a$  and  $b$  are on the same side of  $\mathcal{B}(b, c)$ . Let  $x$  be the reflection of  $a$  over  $\mathcal{B}(b, c)$ . The line  $ax$  is parallel to the line  $bc$ , and  $x$  and  $c$  are on the same side of  $ab$ . Hence,  $x$  is in the interior of the cone defined by  $\angle abc$ , and hence  $\angle xbc$  is less than  $\angle abc$ . Since  $c$  was chosen so that  $\angle abc$  is minimal,  $x \notin \mathcal{P}$ , which completes the proof.  $\square$

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## 2 Proof of Theorem 2

The remainder of the paper is devoted to the proof of Theorem 2.

**Handling heavy circles.** We first apply a separate, elementary argument to handle the case that a single circle contains a substantial portion of the points of  $\mathcal{P}$ .

**Lemma 5.** *If a single line or circle contains exactly  $\varepsilon n$  points of  $\mathcal{P}$ , then*

$$|\mathcal{B}| \geq \min(\varepsilon, 1 - \varepsilon) \cdot \varepsilon n^2 / 4.$$

We rely on the following elementary geometric

**Lemma 6.** *Let  $C$  be a circle or a line, and let  $p, q \notin C$  with  $p \neq q$ . Then,*

$$\#\{(r, s) \in C \times C : \mathcal{B}(p, r) = \mathcal{B}(q, s)\} \leq 2.$$

*Proof.* Fix  $p, q \notin C$ . For  $r \in C$ , let  $C_r$  be the reflection of  $C$  over  $\mathcal{B}(p, r)$ ; note that  $C_r = C_{r'}$  implies  $r = r'$ . If  $s \in C$  such that  $\mathcal{B}(p, r) = \mathcal{B}(q, s)$ , then  $q \in C_r$ . Since there are two circles that are the same size as  $C$  and that contain  $p$  and  $q$ , there are at most two pairs  $(r, s) \in C \times C$  such that  $\mathcal{B}(p, r) = \mathcal{B}(q, s)$ .  $\square$

*Proof of Lemma 5.* Let  $C$  be a circle that contains  $\varepsilon n$  points of  $\mathcal{P}$ . Let  $\mathcal{P}' \subset \mathcal{P}$  be a set of  $k = \min(\varepsilon, 1 - \varepsilon)n$  points that are not in  $C$ . Let  $p_1, p_2, \dots, p_k$  be an arbitrary ordering of the points of  $\mathcal{P}'$ . Then, by Lemma 6,  $p_i$  determines a set  $\mathcal{B}(p_i)$  of at least  $\varepsilon n - 2(i - 1)$  distinct perpendicular bisectors with the points of  $P$  that lie on  $C$ , such that no element of  $\mathcal{B}(p_i)$  is an element of  $\mathcal{B}(p_j)$  for any  $j < i$ . Summing over  $i$ , we have

$$\sum_{i \leq k} |\mathcal{B}(p_i)| \geq \varepsilon nk/4,$$

which proves the lemma.  $\square$

Lemma 5 implies that, if the maximum number of points of  $\mathcal{P}$  that are contained in any circle is at least  $cn^2$  (for a  $c$  to be determined later) and at most  $(1 - O(n^{-18/35+\epsilon}))n = (1 - \delta)n$  points, then  $|\mathcal{B}| = \Omega(n^{52/35-\varepsilon})$ . Hence, we may assume from now on that no circle contains more than  $cn^2$  points.

**Refining the pairs of points.** Now we handle the case that no circle contains more than some small, constant fraction of the points of  $P$ .

Denote by  $s_k$  the number of lines and circles that contain at least  $k$  points of  $P$ , denote by  $s_{=k}$  the number of lines and circles that contain exactly  $k$  points of  $P$ .

The following incidence bound was proved in [1] for circles, and a stronger result for lines was proved by Szemerédi and Trotter [14].

**Lemma 7** (Point-circle incidence bound). *For any  $\varepsilon > 0$ ,*

$$s_k = O(n^{3+\varepsilon}k^{-11/2} + n^2k^{-3} + nk^{-1}),$$

*where the hidden constants depend on  $\varepsilon$ .*

We apply Lemma 7 to show that either a single circle contains a constant fraction of the points of  $\mathcal{P}$ , or a constant fraction of the pairs of points in  $\mathcal{P} \times \mathcal{P}$  are not contained in any circle that contains too many points of  $\mathcal{P}$ .

**Lemma 8.** *For any  $\varepsilon > 0$ , there are constants  $c_1, c_2 > 0$  such that the following holds. Let  $\mathcal{P}$  be a set of  $n$  points. Let  $\Pi \subset \mathcal{P} \times \mathcal{P}$  be the set of pairs of distinct points of  $\mathcal{P}$  such that no pair in  $\Pi$  is contained in a line or a circle that contains  $M = c_1n^{2/7+\varepsilon}$  points of  $\mathcal{P}$ . Then, either there is a single line or circle that contains  $c_2n$  points of  $\mathcal{P}$ , or  $|\Pi| = \Omega(n^2)$ .*

*Proof.* By Lemma 7, the number of triples  $(p, q, C)$  of two points  $p, q$  and a line or a circle  $C$  such that  $p, q \in C$  and such that  $C$  contains at least  $M$  and at

most  $U = c_2 n$  points is bounded above by

$$\begin{aligned}
\sum_{k=M}^U k^2 s_k &= \sum_{k=M}^U k^2 (s_k - s_{k+1}) \\
&\leq \sum_{k=M}^U 2k s_k \\
&\leq O \left( \sum_{k \geq M} n^{3+\varepsilon} k^{-9/2} + \sum_{k \geq M} n^2 k^{-2} + \sum_{k \leq U} n \right) \\
&\leq O(n^2).
\end{aligned}$$

With appropriate choices of  $c_1, c_2$ , we can ensure that the constant hidden in the  $O$ -notation on the final line is less than 1.  $\square$

Let  $\Pi \subseteq \mathcal{P} \times \mathcal{P}$  be the set of pairs of distinct points of  $\mathcal{P}$  that do not lie on any circle that contains more than  $M = c_1 n^{2/7+\varepsilon}$  points of  $\mathcal{P}$ . We have already handled the case that a single line or circle contains  $c_2 n$  points of  $\mathcal{P}$ , and so we assume that this does not occur, and consequently (by Lemma 8) that  $|\Pi| = \Omega(n^2)$ .

Let

$$\begin{aligned}
\mathcal{B}^* &= \{\mathcal{B}(a, b) : (a, b) \in \Pi\}, \text{ and} \\
\mathcal{Q}^* &= \{(a, b, c, d) : (a, b), (c, d) \in \Pi, \mathcal{B}(a, b) = \mathcal{B}(c, d)\}.
\end{aligned}$$

An application of Cauchy-Swartz produces a lower bound on  $|\mathcal{B}^*|$  (and hence, on  $|\mathcal{B}|$ ) from an upper bound on  $|\mathcal{Q}^*|$ .

**Lemma 9.**

$$|\mathcal{B}| = \Omega(n^4 |\mathcal{Q}^*|^{-1}).$$

*Proof.* For a line  $\ell$ , denote by  $w(\ell)$  the number of pairs  $(p, q) \in \Pi$  such that  $\mathcal{B}(p, q) = \ell$ . By Cauchy-Schwartz,

$$\begin{aligned}
|\mathcal{Q}^*| &= \sum_{\ell \in \mathcal{B}^*} w(\ell)^2, \\
&\geq \left( \sum_{\ell \in \mathcal{B}^*} w(\ell) \right)^2 |\mathcal{B}^*|^{-1}, \\
&= |\Pi|^2 |\mathcal{B}^*|^{-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\mathcal{B}| &\geq |\mathcal{B}^*|, \\
&\geq |\Pi|^2 |\mathcal{Q}^*|^{-1}, \\
&= \Omega(n^4 |\mathcal{Q}^*|^{-1}).
\end{aligned}$$

$\square$

**Bounding the energy.** We will use another incidence geometry argument to bound  $|Q^*|$ ; this part of the analysis has substantial overlap with the proof of Theorem 2.1 in [11].

For each pair  $(a, b) \in P^2$ , let  $C(a, b)$  be the maximum number of points on any circle that contains  $a, b$ . Let

$$\begin{aligned}\Pi_K &= \{(a, b) \in P^2 : a \neq b, C(a, b) \leq K\}, \\ \mathcal{Q}_K &= \{(a, b, c, d) \in P^4 : (a, b), (c, d) \in \Pi_K, \mathcal{B}(a, b) = \mathcal{B}(c, d)\}.\end{aligned}$$

We prove

**Lemma 10.** *For any  $2 \leq K \leq n$ ,*

$$|Q_K| = O\left(K^{\frac{2}{5}} n^{\frac{12}{5} + \varepsilon} + Kn^2\right).$$

*Proof.* For each pair  $(a, c)$  of distinct points in  $\mathcal{P}$ , we define the *bisector surface* to be

$$S_{ac} = \{(b, d) \in \mathbb{R}^2 \times \mathbb{R}^2 : \mathcal{B}(a, b) = \mathcal{B}(c, d)\},$$

and we define

$$\mathcal{S} = \{S_{ac} : a, c \in \mathcal{P}, a \neq c\}.$$

The following is [11, Lemma 3.1].

**Lemma 11.** *For distinct  $a, c \in \mathcal{P}$ , there exists a two-dimensional constant-degree algebraic variety  $\overline{S}_{ac}$  such that  $S_{ac} \subset \overline{S}_{ac}$ . Moreover, if  $(b, d) \in (\overline{S}_{ac} \setminus S_{ac})$  with  $b \neq d$ , then either  $a = b$  or  $c = d$ .*

We denote

$$\overline{\mathcal{S}} = \{\overline{S}_{ac} : a, c \in \mathcal{P}, a \neq c\}.$$

Let  $G \subset \overline{\mathcal{S}} \times \mathcal{P}^2$  be the incidence graph between pairs of distinct points of  $\mathcal{P}$  and varieties in  $\overline{\mathcal{S}}$ . Let  $H \subset \mathcal{S} \times \mathcal{P}^2$  be the incidence graph between pairs of distinct points of  $\mathcal{P}$  and surfaces in  $\mathcal{S}$ . Let  $G' \subset \mathcal{S} \times \mathcal{P}^2$  such that  $(S_{ac}, (b, d)) \in G'$  if and only if  $(b, d) \in S_{ac}$  and  $(a, b), (c, d) \in \Pi_K$ . By identifying the vertices corresponding to  $S_{ac}$  and  $\overline{S}_{ac}$  for each  $a, c$ , we have  $G' \subseteq H \subseteq G$ .

Note that

$$G' = \{(S_{ac}, (b, d)) \in \mathcal{S} \times \mathcal{P}^2 : (a, b), (c, d) \in \Pi_K, B(a, c) = B(b, d)\},$$

and hence

$$|G'| = |Q^*|.$$

Observe that if  $\mathcal{B}(a, b) = \mathcal{B}(c, d)$ , then the reflection of the pair  $(a, c)$  over the line  $\mathcal{B}(a, b)$  is the pair  $(b, d)$ . Hence,  $|ac| = |bd|$ . It follows that, if  $|ac| = \delta$ , then the surface  $S_{ac}$  is contained in the hypersurface

$$H_\delta = \{(b, d) \in \mathbb{R}^2 \times \mathbb{R}^2 : |bd| = \delta\}.$$

The following is [11, Lemma 3.2].

**Lemma 12.** *Let  $a, c \in \mathbb{R}^2$ ,  $(a, c) \neq (a', c')$  and  $|ac| = |a'c'| = \delta \neq 0$ . Then there exist curves  $C_1, C_2 \subset \mathbb{R}^2$ , which are either two concentric circles or two parallel lines, such that  $a, a' \in C_1$ ,  $c, c' \in C_2$ , and  $S_{ac} \cap S_{a'c'}$  is contained in the set*

$$H_\delta \cap (C_1 \times C_2) = \{(b, d) \in \mathbb{R}^2 \times \mathbb{R}^2 : b \in C_1, d \in C_2, |bd| = \delta\}.$$

We use Lemmas 11 and 12 to prove

**Lemma 13.** *If  $\overline{S}_{ac}, \overline{S}_{a'c'}$  have  $K + 4$  or more common neighbors in  $G$ , then  $S_{ac}, S_{a'c'}$  have no common neighbors in  $G'$ .*

*Proof.* We claim that, if  $S_{ab}$  and  $S_{a'b'}$  share  $K$  or more common neighbors in  $H$ , then they have no common neighbors in  $G'$ . Suppose that  $S_{ac}$  and  $S_{a'c'}$  share  $K$  common neighbors in  $H$ ; denote this set of pairs of points  $(b, d) \in S_{ac} \cap S_{a'c'}$  by  $\Gamma$ . Lemma 12 implies that there exist two lines or circles  $C_1, C_2$  with  $a, a' \in C_1$  and  $c, c' \in C_2$  such that  $(b, d) \in \Gamma$  only if  $b \in C_1$  and  $d \in C_2$ , and  $|bd| = \delta$ . Note that  $(b, d), (b, d') \in S_{ac}$  implies that  $\mathcal{B}(c, d) = \mathcal{B}(a, b) = \mathcal{B}(c, d')$ , and hence  $d = d'$ . Hence, there are at least  $K$  distinct points in each of  $C_1$  and  $C_2$ , and so, if  $(b, d) \in \Gamma$ , then  $(a, b), (a', b), (c, d), (c', d) \notin \Pi_K$ . Hence, if  $(b, d) \in \Gamma$ , then  $(S_{ac}, (b, d))$  and  $(S_{a'c'}, (b, d))$  are not in  $G'$ .

Suppose that  $a \neq a'$  and  $c \neq c'$ . By Lemma 11, each neighbor of  $\overline{S}_{ac}$  in  $G \setminus H$  is either of the form  $(a, d)$  or  $(b, c)$ . Hence, a vertex  $(b, d)$  can be a common neighbor of  $S_{ac}$  and  $S_{a'c'}$  in  $G$  but not in  $H$  only if  $b = a$  or  $b = a'$  or  $d = c$  or  $d = c'$ . Since  $S_{ac}$  is incident to at most one point  $(b, x)$  for any fixed  $b$ , this implies that  $S_{ac}, S_{a'c'}$  have at most 4 common neighbors in  $G$  that are not common neighbors in  $H$ . Hence, if  $\overline{S}_{ac}$  and  $\overline{S}_{a'c'}$  have  $K + 4$  or more common neighbors in  $G$ , then  $S_{ac}$  and  $S_{a'c'}$  have  $K$  or more common neighbors in  $H$ , and by the previous claim have no common neighbors in  $G'$ .

Now, suppose that  $a = a'$  (the case  $c = c'$  is symmetric). Since  $(b, d) \in S_{ac} \cap S_{a'c'}$  implies that  $c = c'$ , we have that  $S_{ac}$  and  $S_{a'c'}$  have no common neighbors in  $H$ . Hence, they have no common neighbors in  $G'$ .  $\square$

Let  $\delta_1, \dots, \delta_D$  denote the distinct non-zero distances determined by pairs of distinct points in  $\mathcal{P}$ . Let

$$\begin{aligned} \mathcal{P}_i^2 &= \{(b, d) \in \mathcal{P} \times \mathcal{P} : |ab| = \delta_i\}, \\ \mathcal{S}_i &= \{S_{ac} \in \mathcal{S} : |ac| = \delta_i\}, \\ G'_i &= \{(S_{ac}, (b, d)) \in G' : |pq| = \delta_i\}. \end{aligned}$$

Let

$$m_i = |\mathcal{P}_i^2| = |\mathcal{S}_i|.$$

As observed above, each quadruple  $(a, b, c, d) \in Q$  satisfies  $|ac| = |bd|$ . Hence, it suffices to study each  $G'_i$  separately. That is, we have

$$|Q_K| = |G'| = \sum_{i=1}^D |G'_i|.$$



We will use the following incidence bound, proved in section 6, to control the size of each  $|G'_i|$ . This bound is a slight generalization of a bound in [11], which is in turn a slight generalization of a bound in [5]. See [5] for definitions of the algebraic terms used.

**Theorem 14.** *Let  $\mathcal{S}$  be a set of  $n$  constant-degree varieties, and let  $\mathcal{P}$  be a set of  $m$  points, both in  $\mathbb{R}^d$ . Let  $s \geq 2$  be a constant, and  $t \geq 2$  be a function of  $m, n$ . Let  $G$  be the incidence graph of  $\mathcal{P} \times \mathcal{S}$ . Let  $G' \subseteq G$  such that, if a set  $L$  of  $s$  left vertices has a common neighborhood of size  $t$  or more in  $G$ , then no pair of vertices in  $L$  has a common neighbor in  $G'$ . Moreover, suppose that  $\mathcal{P} \subset V$ , where  $V$  is an irreducible constant-degree variety of dimension  $e$ . Then*

$$|G'| = O\left(m^{\frac{s(e-1)}{es-1}+\varepsilon} n^{\frac{e(s-1)}{es-1}} t^{\frac{e-1}{es-1}} + tm + n\right).$$

We apply Theorem 14 to the set of varieties  $\overline{\mathcal{S}}_i = \{\overline{S}_{ac} : S_{ac} \in \mathcal{S}_i\}$ , the set of points  $\mathcal{P}_i^2$ , and  $G$  and  $G'$  as the corresponding incidence graphs. The hypersurface  $H_{\delta_i}$  is irreducible, three-dimensional, and of constant degree, since it is defined by the irreducible polynomial  $(x_1 - x_3)^2 + (x_2 - x_4)^2 - \delta_i$ . Thus, we can apply Theorem 14 with  $m = n = m_i$ ,  $V = H_{\delta_i}$ ,  $d = 4$ ,  $e = 3$ ,  $s = 2$ , and  $t = K$ . Hence,

$$|G'_i| = O(K^{2/5} m_i^{7/5+\varepsilon} + K m_i). \quad (3)$$

Let  $J$  be the set of indices  $1 \leq j \leq D$  for which the bound in (3) is dominated by the term  $K^{\frac{2}{5}} m_j^{\frac{7}{5}+\varepsilon}$ . By recalling that  $\sum_{j=1}^D m_j = n(n-1)$ , we get

$$\sum_{j \notin J} |G'_j| = O(K n^2).$$

Next we consider  $\sum_{j \in J} |G'_j| = O(\sum_{j \in J} K^{2/5} m_j^{7/5+\varepsilon})$ . By [6, Proposition 2.2], we have

$$\sum m_j^2 = O(n^3 \log n).$$

This implies that the number of  $m_j$  for which  $m_j \geq x$  is  $O(n^3 \log n / x^2)$ . Using a dyadic decomposition, we obtain

$$\begin{aligned} K^{-2/5} n^{-\varepsilon} \sum_{j \in J} |G'_j| &= O\left(\sum_{m_j \leq \Delta} m_j^{7/5} + \sum_{k \geq 1} \sum_{2^{k-1} \Delta < m_j \leq 2^k \Delta} m_j^{7/5}\right) \\ &= O\left(\Delta^{7/5} \cdot \frac{n^2}{\Delta} + \sum_{k \geq 1} (2^k \Delta)^{\frac{7}{5}} \cdot \frac{n^3 \log n}{(2^{k-1} \Delta)^2}\right) \\ &= O\left(\Delta^{2/5} n^2 + \frac{n^3 \log n}{\Delta^{3/5}}\right). \end{aligned}$$

By setting  $\Delta = n \log n$ , we have

$$\sum_{j \in J} |G'_j| = O\left(K^{\frac{2}{5}} n^{\frac{12}{5}+\varepsilon} \log^{\frac{2}{5}} n\right) = O\left(K^{\frac{2}{5}} n^{\frac{12}{5}+\varepsilon'}\right).$$

Combining the bounds, we have

$$|\mathcal{Q}^*| = |G'| = \sum_{i=1}^D |G'_i| = O\left(K^{\frac{2}{5}} n^{\frac{12}{5} + \varepsilon'} + Kn^2\right),$$

which completes the proof of Lemma 10.  $\square$

**Finishing the proof.** Taking  $K = M = c_1 n^{2/7+\varepsilon}$  in Lemma 10, we have

$$|\mathcal{Q}^*| = O(n^{88/35+\varepsilon}).$$

Combining this with Lemma 9, we have

$$|\mathcal{B}| = \Omega(n^{52/35-\varepsilon}),$$

which is Theorem 2.

### 3 Proof of Theorem 3

We prove Theorem 3 by double counting the set

$$\Delta = \{(a, b, c) \in P^3 : |ab| = |ac|, b \neq c\},$$

which is the set of oriented, non-degenerate isosceles triangles determined by  $P$ .

The lower bound on  $\Delta$  proceeds by a standard application of the Cauchy-Schwarz inequality. We denote by  $n(p, \delta)$  the number of points of  $P$  at distance  $\delta$  from  $p$ .

$$\begin{aligned} |\Delta| &= \sum_{p \in P} \sum_{\delta \in \delta(p)} (n(p, \delta) - 1)^2, \\ &\geq \sum_{p \in P} (n - |\delta(p)|)^2 |\delta(p)|^{-1}, \\ &\geq \sum_{p \in P} (n - \delta^*)^2 (\delta^*)^{-1}, \\ &= n(n - \delta^*)^2 (\delta^*)^{-1}. \end{aligned}$$

For the upper bound, we apply the observation that the number of isosceles triangles is equal to the number of incidences between the points of  $P$  and bisectors of  $P$ , counted with multiplicity. We denote the multiplicity of a bisector  $\ell \in \mathcal{B}$  by

$$w(\ell) = |\{(a, b) \in P^2 : \mathcal{B}(a, b) = \ell\}|.$$

It is easy to see that

$$|\Delta| = I(P, \mathcal{B}) = \sum_{p \in P} \sum_{\ell \in \mathcal{B}} [p \in \ell] w(\ell).$$

The notation  $[p \in \ell]$  denotes the indicator function that takes value 1 if  $p \in \ell$  and 0 otherwise.

Recall that, for each pair  $(a, b) \in P^2$ , we denote by  $C(a, b)$  the maximum number of points of  $P$  on any circle that contains both  $a$  and  $b$ . Let

$$\begin{aligned}\Pi_{k,K} &= \{(a, b) \in P^2 : a \neq b, k \leq C(a, b) < K\}, \\ \mathcal{B}_{k,K} &= \{\mathcal{B}(a, b) : (a, b) \in \Pi_{k,K}\}, \\ w_{k,K}(\ell) &= |\{(a, b) \in \Pi_{k,K} : \mathcal{B}(a, b) = \ell\}|.\end{aligned}$$

We decompose the incidences as follows:

$$I(P, \mathcal{B}) = I(P, \mathcal{B}_{2,M}) + \sum_{\log M \leq i < \log n} I(P, \mathcal{B}_{2^i, 2^{i+1}}), \quad (4)$$

in which incidences between  $P$  and  $\mathcal{B}_{k,K}$  are weighted by  $w_{k,K}$ , and, as in Section 2,  $M = c_1 n^{2/7+\varepsilon}$ .

We will use Theorem 18, proved in Section 5, to bound the size of those sets of incidences in (4) involving bisector multiplicities at most  $n^{1/2}$ . Applying Theorem 7, we have

$$\sum_{\ell \in \mathcal{B}_{k,K}} w(\ell) = |\Pi_{k,K}| = \min(n^2, O(k^2(n^{3+\varepsilon}k^{-11/2} + n^2k^{-3} + nk^{-1}))). \quad (5)$$

Applying Lemma 10, we have

$$\sum_{\ell \in \mathcal{B}_{k,K}} w(\ell)^2 = |Q_{k,K}| = O(K^{2/5}n^{12/5+\varepsilon} + Kn^2). \quad (6)$$

It is clear that no line can be the bisector of more than  $n$  pairs of points.

Recall that  $Q^* = Q_{2,M}$  and  $\Pi = \Pi_{2,M}$ . Applying Theorem 18 together with (5) and (6), we have

$$I(P, \mathcal{B}_{2,M}) = O(n^{2/3}|Q^*|^{1/3}|\Pi|^{1/3} + n^2) = O(n^{228/105+\varepsilon}). \quad (7)$$

Dividing the remaining range for  $2^i < n^{1/2}$  depending on which terms in (5) and (6) are dominant, straightforward calculations show that

$$I(P, \mathcal{B}_{k,2k}) = O(n^{37/15+\varepsilon'}k^{-31/30}), \quad M \leq k < c_a n^{2/5+\varepsilon}, \quad (8)$$

$$I(P, \mathcal{B}_{k,2k}) = O(n^{32/15+\varepsilon'}k^{-1/5}), \quad c_a n^{2/5+\varepsilon} \leq k < c_b n^{1/2}. \quad (9)$$

For  $k \geq c_b n^{1/2}$ , note that Theorem 7 implies that there are  $O(nk^{-1})$  circles that each contain at least  $k$  points. Let  $\mathcal{C}$  be the set of circles that contain between  $k$  and  $2k$  points, for some  $c_b n^{1/2} < k < n$ . Let  $p$  be an arbitrary point of  $P$ , and let  $C$  be an arbitrary circle in  $\mathcal{C}$ . If  $p$  is the center of  $C$ , then  $p$  is incident to all bisectors determined by pairs of points on  $C$ , which have total multiplicity  $O(k^2)$ . Since  $|\mathcal{C}| = O(nk^{-1})$ , there are at most so many centers of circles in  $\mathcal{C}$ , so the total number of such incidences is  $O(nk) = O(n^2)$ . Otherwise,

$p$  is incident to at most one bisector determined by pairs of points on  $C$ , which has total multiplicity  $O(k)$ . Since  $|\mathcal{C}| = O(nk^{-1})$ , there are  $O(n)$  such incidences between  $p$  and circles of  $\mathcal{C}$ , and so the total number of such incidences is  $O(n^2)$ .

Hence, for  $c_b n^{1/2} \leq k \leq n/2$ , we have

$$I(P, \mathcal{B}_{k,2k}) = O(n^2). \quad (10)$$

Note that the right sides of (8), (9), and (10) are all bounded above by  $O(n^{228/105+\varepsilon})$ . Hence, substituting into (4), we have

$$I(P, \mathcal{B}) = O(n^{228/105+\varepsilon} \log(n)).$$

Absorbing the  $\log(n)$  into  $n^\varepsilon$ , this gives us the upper bound

$$|\Delta| = O(n^{228/105+\varepsilon}).$$

Combining the upper and lower bounds, we have

$$n(n - \delta^*)^2 (\delta^*)^{-1} \leq |\Delta| = O(n^{228/105+\varepsilon}),$$

which implies

$$\delta^* = \Omega(n^{87/105-\varepsilon}).$$

## 4 Discussion

The proofs of Theorems 2 and 3 both depend on Theorem 7 and Lemma 10, and neither of these are tight. Any improvement in the bounds for Theorem 7 or Lemma 10 will immediately translate to corresponding improvements to Theorems 2 and 3. Furthermore, if the following conjectured bounds are proved for Theorem 7 and Lemma 10, we will immediately have nearly tight bounds for Theorems 2 and 3.

Let  $P$  be a set of  $n$  points in the Euclidean plane. Recall that  $c_k$  denotes the maximum number of circles that contain at least  $k$  points of  $P$ .

**Conjecture 15.** *For any  $\varepsilon > 0$  and  $k > n^\varepsilon$ ,*

$$c_k = O(n^2 k^{-3} + n k^{-1}).$$

Recall that  $C(a, b)$  denotes the maximum number of points on any circle that contains  $a, b \in P^2$ , and

$$\begin{aligned} \Pi_K &= \{(a, b) \in P^2 : a \neq b, C(a, b) \leq K\}, \\ \mathcal{Q}_K &= \{(a, b, c, d) \in P^4 : (a, b), (c, d) \in \Pi_K, \mathcal{B}(a, b) = \mathcal{B}(c, d)\}. \end{aligned}$$

**Conjecture 16.**

$$|\mathcal{Q}_K| = O(Kn^2).$$

The proofs of Theorems 2 and 3 in Sections 2 and 3 can easily be adapted to use Conjectures 15 and 16. If Conjectures 15 and 16 were proved, this would give (for any  $\varepsilon > 0$ ) the bounds

$$|\mathcal{B}| = \Omega(n^{2-\varepsilon}), \text{ and } \delta^* = \Omega(n^{1-\varepsilon}),$$

where the lower bound on  $|\mathcal{B}|$  depends on the assumption that no more than  $cn$  points lie on any circle, for some  $c < 1$ . Both of these bounds would be tight up to the  $n^\varepsilon$  factors.

## 5 Weighted Szemerédi-Trotter

In this section, we prove a generalized Szemerédi-Trotter theorem for weighted points and lines, which is useful when we have control over the sum-of-squares of the weights. This strengthens an earlier weighted Szemerédi-Trotter from [9], in the case that the weights of lines or points differ substantially.

For a set  $A$  with a weight function  $w : A \rightarrow \mathbb{Z}^+$ , let

$$\begin{aligned} |A|_1 &= \sum_{a \in A} w(a), \\ |A|_2^2 &= \sum_{a \in A} w(a)^2, \\ |A|_\infty &= \max_{a \in A} w(a). \end{aligned}$$

For a weighted set  $P$  of points and a weighted set  $L$  of lines, define

$$I(P, L) = \sum_{p \in P} \sum_{\ell \in L} [p \in \ell] w(p) w(\ell)$$

to be the number of weighted incidences between  $P$  and  $L$ .

We need the standard Szemerédi-Trotter theorem for unweighted points and lines.

**Theorem 17** (Szemerédi-Trotter). *Let  $P$  be a set of points, and  $L$  a set of lines, in  $\mathbb{R}^2$ . Then,*

$$I(P, L) = O(|P|^{2/3} |L|^{2/3} + |P| + |L|).$$

**Theorem 18** (Weighted Szemerédi-Trotter). *Let  $P$  be a set of weighted points, and  $L$  a set of weighted lines, in  $\mathbb{R}^2$ . Then,*

$$I(P, L) = O\left((|P|_2^2 |P|_1 |L|_2^2 |L|_1)^{1/3} + |L|_\infty |P|_1 + |P|_\infty |L|_1\right).$$

*Proof.* Let

$$\begin{aligned} L_i &= \{\ell \in L : 2^i \leq w(\ell) < 2^{i+1}\}, \\ P_i &= \{p \in P : 2^i \leq w(p) < 2^{i+1}\}. \end{aligned}$$

Then, applying a dyadic decomposition and Theorem 17, we have

$$\begin{aligned} I(P, L) &= \sum_{\ell \in L} \sum_{p \in P} [p \in \ell] w(p) w(\ell), \\ &\ll \sum_{1 \leq 2^i < |L|_\infty} \sum_{1 \leq 2^j < |P|_\infty} 2^{i+1} 2^{j+1} (|L_i|^{2/3} |P_j|^{2/3} + |L_i| + |P_j|), \end{aligned} \quad (11)$$

Since

$$|L|_1 = \sum_{\ell \in L} w(\ell) \geq \sum_{1 \leq 2^i < |L|_\infty} 2^i |L_i|,$$

we have

$$\sum_{1 \leq 2^i < |L|_\infty} \sum_{1 \leq 2^j < |P|_\infty} 2^i 2^j |L_i| \leq |L|_1 \sum_{1 \leq 2^j < |P|_\infty} 2^j \ll |L|_1 |P|_\infty. \quad (12)$$

Similarly,

$$\sum_{1 \leq 2^i < |L|_\infty} \sum_{1 \leq 2^j < |P|_\infty} 2^i 2^j |P_i| \ll |P|_1 |L|_\infty. \quad (13)$$

Next, we bound the term  $\sum_{1 \leq 2^i < |L|_\infty} 2^i |L_i|^{2/3}$  in (11). We split the sum as

$$\sum_{1 \leq 2^i < |L|_\infty} 2^i |L_i|^{2/3} = \sum_{1 \leq 2^i < |L|_2^2 |L|_1^{-1}} 2^i |L_i|^{2/3} + \sum_{|L|_2^2 |L|_1^{-1} \leq 2^i < |L|_\infty} 2^i |L_i|^{2/3}.$$

Note that  $\sum_i 2^i |L_i| \leq |L|_1$ , and hence  $2^i |L_i| \leq |L|_1$  for any particular  $i$ . Also note that  $\sum_{1 \leq 2^i < |L|_2^2 |L|_1^{-1}} 2^{i/3}$  is only a constant factor larger than its largest term,  $|L|_2^{2/3} |L|_1^{-1/3}$ .

$$\begin{aligned} \sum_{1 \leq 2^i < |L|_2^2 |L|_1^{-1}} 2^i |L_i|^{2/3} &\leq \sum_{1 \leq 2^i < |L|_2^2 |L|_1^{-1}} 2^i (|L|_1 2^{-i})^{2/3}, \\ &= |L|_1^{2/3} \sum_{1 \leq 2^i < |L|_2^2 |L|_1^{-1}} 2^{i/3}, \\ &\ll |L|_1^{1/3} |L|_2^{2/3}. \end{aligned}$$

Note that  $\sum_i 2^{2i} |L_i| \leq |L|_2^2$ , and hence  $2^{2i} |L_i| \leq |L|_2^2$  for any particular  $i$ .

$$\begin{aligned} \sum_{|L|_2^2 |L|_1^{-1} \leq 2^i < |L|_\infty} 2^i |L_i|^{2/3} &\leq \sum_{|L|_2^2 |L|_1^{-1} \leq 2^i < |L|_\infty} 2^i (|L|_2^2 2^{-2i})^{2/3}, \\ &= |L|_2^{4/3} \sum_{|L|_2^2 |L|_1^{-1} \leq 2^i < |L|_\infty} 2^{-i/3}, \\ &\ll |L|_1^{1/3} |L|_2^{2/3}. \end{aligned}$$

Hence,

$$\sum_{1 \leq 2^i < |L|_\infty} |L_i|^{2/3} \ll |L|_1^{1/3} |L|_2^{2/3}, \quad (14)$$

and similarly,

$$\sum_{1 \leq 2^j < |P|_\infty} |P_j|^{2/3} \ll |P|_1^{1/3} |P|_2^{2/3}. \quad (15)$$

Combining (12), (13), (14), and (15) with (11) completes the proof.  $\square$

## 6 Proof of Theorem 14

The proof of Theorem 14 is nearly identical to the proof of Theorem 2.5 in [11]. The main difference occurs in bounding the quantity  $|I_1|$  (defined below).

The proof uses the Kővári-Sós-Turán theorem (see for example [2, Theorem IV.9]).

**Lemma 19** (Kővári-Sós-Turán). *Let  $G$  be a bipartite graph with vertex set  $A \cup B$ . Let  $s \leq t$ . Suppose that  $G$  contains no  $K_{s,t}$ ; that is, for any  $s$  vertices in  $A$ , at most  $t - 1$  vertices in  $B$  are connected to each of the  $s$  vertices. Then*

$$|G| = O(t^{\frac{1}{s}} |A| |B|^{\frac{s-1}{s}} + |B|).$$

We amplify the weak bound of Lemma 19 by using *polynomial partitioning*. Given a polynomial  $f \in \mathbb{R}[x_1, \dots, x_d]$ , we write  $Z(f) = \{p \in \mathbb{R}^d : f(p) = 0\}$ . We say that  $f \in \mathbb{R}[x_1, \dots, x_d]$  is an  *$r$ -partitioning polynomial* for a finite set  $\mathcal{P} \subset \mathbb{R}^d$  if no connected component of  $\mathbb{R}^d \setminus Z(f)$  contains more than  $|\mathcal{P}|/r$  points of  $\mathcal{P}$  (notice that there is no restriction on the number of points of  $\mathcal{P}$  that are in  $Z(f)$ ). Guth and Katz [6] introduced this notion and proved that for every  $\mathcal{P} \subset \mathbb{R}^d$  and  $1 \leq r \leq |\mathcal{P}|$ , there exists an  $r$ -partitioning polynomial of degree  $O(r^{1/d})$ . In [5], the following generalization was proved.

**Theorem 20** (Partitioning on a variety). *Let  $V$  be an irreducible variety in  $\mathbb{R}^d$  of dimension  $e$  and degree  $D$ . Then for every finite  $\mathcal{P} \subset V$  there exists an  $r$ -partitioning polynomial  $f$  of degree  $O(r^{1/e})$  such that  $V \not\subset Z(f)$ . The implicit constant depends only on  $d$  and  $D$ .*

We are now ready to prove the incidence bound.

*Proof of Theorem 14.* Note that we may assume that no variety in  $\mathcal{S}$  contains  $V$ . We can assume that  $V$  contains at least  $s$  points (otherwise the bound in the theorem is trivial). If there are at most  $t - 1$  varieties in  $\mathcal{S}$  that contain  $V$ , then these varieties altogether give less than  $tm$  incidences, which is accounted for in the bound. If there are  $t$  or more varieties in  $\mathcal{S}$  that contain  $V$ , then Lemma 13 implies that no pair of vertices in  $G'$  corresponding to a pair of points contained in  $V$  shares any neighbor among the vertices corresponding to the varieties that contain  $V$ . These are at most  $m$  incidences, which is accounted for in the bound.

We use induction on  $e$  and  $m$ , with the induction claim being that for  $\mathcal{P}, \mathcal{S}, V, G'$  as in the theorem, with the added condition that no variety in  $\mathcal{S}$  contains  $V$ , we have

$$|G'| \leq \alpha_{1,e} m^{\frac{s(e-1)}{es-1} + \varepsilon} n^{\frac{e(s-1)}{es-1}} t^{\frac{e-1}{es-1}} + \alpha_{2,e}(tm + n), \quad (16)$$

for constants  $\alpha_{1,e}, \alpha_{2,e}$  depending only on  $d, e, s, \varepsilon$ , the degree of  $V$ , and the degrees of the varieties in  $\mathcal{S}$ . The base cases for the induction are simple. If  $m$  is sufficiently small, then (16) follows immediately by choosing sufficiently large values for  $\alpha_{1,e}$  and  $\alpha_{2,e}$ . Similarly, when  $e = 0$ , we again obtain (16) when  $\alpha_{1,e}$  and  $\alpha_{2,e}$  are sufficiently large (as a function of  $d$  and the degree of  $V$ ).

The constants  $d, e, s, \varepsilon$  are given and thus fixed, as are the degree of  $V$  and the degrees of the varieties in  $\mathcal{S}$ . The other constants are to be chosen, and the dependencies between them are

$$C_{\text{weak}}, C_{\text{part}}, C_{\text{inter}} \ll C_{\text{cells}} \ll C_{\text{Höld}} \ll r \ll C_{\text{comps}} \ll \alpha_{2,e} \ll \alpha_{1,e},$$

where  $C \ll C'$  means that  $C'$  is to be chosen sufficiently large compared to  $C$ ; in particular,  $C$  should be chosen before  $C'$ . Furthermore, the constants  $\alpha_{1,e}, \alpha_{2,e}$  depend on  $\alpha_{1,e-1}, \alpha_{2,e-1}$ .

Note that  $G'$  is  $K_{s,t}$ -free. Hence, by Lemma 19, there exists a constant  $C_{\text{weak}}$  depending on  $d, s$  such that

$$|G'| \leq C_{\text{weak}} \left( mn^{1-\frac{1}{s}} t^{\frac{1}{s}} + n \right).$$

When  $m \leq (n/t)^{1/s}$ , and  $\alpha_{2,e}$  is sufficiently large, we have  $|G'| \leq \alpha_{2,e}n$ . Therefore, in the remainder of the proof we can assume that  $n < m^s t$ , which implies

$$n = n^{\frac{e-1}{es-1}} n^{\frac{e(s-1)}{es-1}} \leq m^{\frac{s(e-1)}{es-1}} n^{\frac{e(s-1)}{es-1}} t^{\frac{(e-1)}{es-1}}. \quad (17)$$

**Partitioning.** By Theorem 20, there exists an  $r$ -partitioning polynomial  $f$  with respect to  $V$  of degree at most  $C_{\text{part}} \cdot r^{1/e}$ , for a constant  $C_{\text{part}}$ . Denote the cells of  $V \setminus Z(f)$  as  $\Omega_1, \dots, \Omega_N$ . Since we are working over the reals, there exists a constant-degree polynomial  $g$  such that  $Z(g) = V$ . Then, by [12, Theorem A.2], the number of cells is bounded by  $C \cdot \deg(f)^{\dim V} = C_{\text{cells}} \cdot r$ , for some constant  $C_{\text{cells}}$  depending on  $C_{\text{part}}$ .

We partition  $G'$  into the following three subsets:

- $I_1$  consists of the incidences  $(p, S) \in \mathcal{P} \times \mathcal{S}$  such that  $p \in V \cap Z(f)$ , and some irreducible component of  $V \cap Z(f)$  contains  $p$  and is fully contained in  $S$ .
- $I_2$  consists of the incidences  $(p, S) \in \mathcal{P} \times \mathcal{S}$  such that  $p \in V \cap Z(f)$ , and no irreducible component of  $V \cap Z(f)$  that contains  $p$  is contained in  $S$ .
- $I_3 = G' \setminus (I_1 \cup I_2)$ , the set of incidences  $(p, S) \in \mathcal{P} \times \mathcal{S}$  such that  $p$  is not contained in  $V \cap Z(f)$ .

Note that we indeed have  $G' = I_1 \cup I_2 \cup I_3$ , excluding any edges corresponding to varieties in  $\mathcal{S}$  that fully contain  $V$ .



**Bounding  $|I_1|$ .** The points of  $\mathcal{P} \subset \mathbb{R}^d$  that participate in incidences of  $I_1$  are all contained in the variety  $V_0 = V \cap Z(f)$ . Set  $\mathcal{P}_0 = \mathcal{P} \cap V_0$  and  $m_0 = |\mathcal{P}_0|$ . Since  $V$  is an irreducible variety and  $V \not\subset Z(f)$ ,  $V_0$  is a variety of dimension at most  $e-1$  and of degree that depends on  $r$ . By [12, Lemma 4.3], the intersection  $V_0$  is a union of  $C_{\text{comps}}$  irreducible components, where  $C_{\text{comps}}$  is a constant depending on  $r$  and  $d$ .<sup>3</sup> The degrees of these components also depend only on these values (for a proper definition of degrees and further discussion, see for instance [5]).

Consider an irreducible component  $W$  of  $V_0$ . If  $W$  contains at most  $s-1$  points of  $\mathcal{P}_0$ , it yields at most  $(s-1)n$  incidences. Otherwise, if there are at most  $t-1$  varieties of  $\mathcal{S}$  that fully contain  $W$ , then these yield at most  $(t-1)m_0$  incidences. Otherwise, if there are at least  $t$  varieties of  $\mathcal{S}$  that fully contain  $W$ , then by Lemma 13, no pair of vertices corresponding to points contained in  $W$  has a common neighbor in  $G'$  among the varieties that contain  $W$ . In this case, at most  $m_0$  incidences must be counted.

By summing up, choosing sufficiently large  $\alpha_{1,e}, \alpha_{2,e}$ , and applying (17), we have

$$|I_1| \leq C_{\text{comps}}(sn + tm_0) < \frac{\alpha_{2,e}}{2}(n + tm_0) < \frac{\alpha_{1,e}}{4} m^{\frac{s(e-1)}{e-1}} n^{\frac{e(s-1)}{e-1}} t^{\frac{(e-1)}{e-1}} + \frac{\alpha_{2,e}}{2} tm_0. \quad (18)$$

**Bounding  $|I_2|$ .** The points that participate in  $I_2$  lie in  $V_0 = V \cap Z(f)$ , and the varieties that participate do not contain any component of  $V_0$ . Because  $V_0$  has dimension at most  $e-1$ , we can apply the induction claim on each irreducible component  $W$  of  $V_0$ , for the point set  $\mathcal{P} \cap W$  and the set of varieties in  $\mathcal{S}$  that do not contain  $W$ . Since  $V_0$  has  $C_{\text{comps}}$  irreducible components, we get

$$|I_2| \leq C_{\text{comps}} \alpha_{1,e-1} m_0^{\frac{s(e-2)}{(e-1)s-1} + \varepsilon} n^{\frac{(e-1)(s-1)}{(e-1)s-1}} t^{\frac{e-2}{(e-1)s-1}} + \alpha_{2,e-1}(tm_0 + n),$$

with  $\alpha_{1,e-1}$  and  $\alpha_{2,e-1}$  depending on the degree of the irreducible component of  $V_0$ , which in turn depends on  $r$ . Recalling that we may assume  $n < m^s t$ , we obtain

$$\begin{aligned} m^{\frac{s(e-2)}{(e-1)s-1} + \varepsilon} n^{\frac{(e-1)(s-1)}{(e-1)s-1}} t^{\frac{e-2}{(e-1)s-1}} &= m^{\frac{s(e-2)}{(e-1)s-1} + \varepsilon} n^{\frac{e(s-1)}{e-1}} n^{\frac{s-1}{(es-s-1)(e-1)}} t^{\frac{e-2}{(e-1)s-1}} \\ &< m^{\frac{s(e-1)}{e-1} + \varepsilon} n^{\frac{e(s-1)}{e-1}} t^{\frac{e-1}{e-1}}. \end{aligned}$$

By applying (17) to remove the term  $\alpha_{2,e-1}n$ , and by choosing  $\alpha_{1,e}$  and  $\alpha_{2,e}$  sufficiently large as a function of  $C_{\text{comps}}, \alpha_{1,e-1}, \alpha_{2,e-1}$ , we obtain

$$|I_2| \leq \frac{\alpha_{1,e}}{4} m^{\frac{s(e-1)}{e-1} + \varepsilon} n^{\frac{e(s-1)}{e-1}} t^{\frac{e-1}{e-1}} + \frac{\alpha_{2,e}}{2} tm_0. \quad (19)$$

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<sup>3</sup>This lemma only applies to complex varieties. However, we can take the *complexification* of the real variety and apply the lemma to it (for the definition of a complexification, see for example [15, Section 10]). The number of irreducible components of the complexification cannot be smaller than number of irreducible components of the real variety (see for instance [15, Lemma 7]).

**Bounding  $|I_3|$ .** For every  $1 \leq i \leq N$ , we set  $\mathcal{P}_i = \mathcal{P} \cap \Omega_i$  and denote by  $\mathcal{S}_i$  the set of varieties of  $\mathcal{S}$  that intersect the cell  $\Omega_i$ . Let  $G'_i \subseteq G'$  be  $(\mathcal{P}_i \times \mathcal{S}_i) \cap G'$ . We also set  $m_i = |\mathcal{P}_i|$  and  $n_i = |\mathcal{S}_i|$ . Then we have  $m_i \leq m/r$  and  $\sum_{i=1}^N m_i = m - m_0$ .

Let  $S \in \mathcal{S}$ . By the assumption made at the beginning of the proof,  $S$  does not contain  $V$ , so  $S \cap V$  is a subvariety of  $V$  of dimension at most  $e - 1$ . By [12, Theorem A.2], there exists a constant  $C_{\text{inter}}$  such that the number of cells intersected by  $S \cap V$  is at most  $C \cdot \deg(f)^{\dim(S \cap V)} = C_{\text{inter}} \cdot r^{(e-1)/e}$ . This implies that

$$\sum_{i=1}^N n_i \leq C_{\text{inter}} \cdot r^{\frac{e-1}{e}} \cdot n.$$

By Hölder's inequality we have

$$\begin{aligned} \sum_{i=1}^N n_i^{\frac{e(s-1)}{es-1}} &\leq \left( \sum_{i=1}^N n_i \right)^{\frac{e(s-1)}{es-1}} \left( \sum_{i=1}^N 1 \right)^{\frac{e-1}{es-1}} \\ &\leq \left( C_{\text{inter}} r^{(e-1)/e} n \right)^{\frac{e(s-1)}{es-1}} (C_{\text{cells}} r)^{\frac{e-1}{es-1}} \\ &\leq C_{\text{Höld}} r^{\frac{(e-1)s}{es-1}} n^{\frac{e(s-1)}{es-1}}, \end{aligned}$$

where  $C_{\text{Höld}}$  depends on  $C_{\text{inter}}, C_{\text{cells}}$ . Using the induction claim for each  $i$  with the point set  $\mathcal{P}_i$ , the set of varieties  $\mathcal{S}_i$ , and the same variety  $V$ , we obtain

$$\begin{aligned} \sum_{i=1}^N |G'_i| &\leq \sum_{i=1}^N \left( \alpha_{1,e} m_i^{\frac{(e-1)s}{es-1} + \varepsilon} n_i^{\frac{e(s-1)}{es-1}} t^{\frac{(e-1)}{es-1}} + \alpha_{2,e} (tm_i + n_i) \right) \\ &\leq \alpha_{1,e} \frac{m^{\frac{(e-1)s}{es-1} + \varepsilon} t^{\frac{(e-1)}{es-1}}}{r^{\frac{(e-1)s}{es-1} + \varepsilon}} \sum_{i=1}^N n_i^{\frac{e(s-1)}{es-1}} + \sum_{i=1}^N \alpha_{2,e} (tm_i + n_i) \\ &\leq \alpha_{1,e} C_{\text{Höld}} \frac{m^{\frac{(e-1)s}{es-1} + \varepsilon} n^{\frac{e(s-1)}{es-1}} t^{\frac{(e-1)}{es-1}}}{r^\varepsilon} + \alpha_{2,e} \left( t(m - m_0) + C_{\text{inter}} r^{\frac{e-1}{e}} n \right). \end{aligned}$$

By choosing  $\alpha_{1,e}$  sufficiently large with respect to  $C_{\text{inter}}, r, \alpha_{2,e}$ , and using (17), we get

$$\sum_{i=1}^N |G'_i| \leq 2\alpha_{1,e} C_{\text{Höld}} \frac{m^{\frac{(e-1)s}{es-1} + \varepsilon} n^{\frac{e(s-1)}{es-1}} s^{\frac{(e-1)}{es-1}}}{r^\varepsilon} + \alpha_{2,e} t(m - m_0).$$

Finally, choosing  $r$  sufficiently large with respect to  $C_{\text{Höld}}$  gives

$$|I_3| = \sum_{i=1}^N I(\mathcal{P}_i, \mathcal{S}_i) \leq \frac{\alpha_{1,e}}{2} m^{\frac{(e-1)s}{es-1} + \varepsilon} n^{\frac{e(s-1)}{es-1}} t^{\frac{(e-1)}{es-1}} + \alpha_{2,e} t(m - m_0). \quad (20)$$

**Summing up.** By combining  $|G'| = |I_1| + |I_2| + |I_3|$  with (18), (19), and (20), we obtain

$$|G'| \leq \alpha_{1,e} m^{\frac{s(e-1)}{es-1} + \varepsilon} n^{\frac{e(s-1)}{es-1}} t^{\frac{(e-1)}{es-1}} + \alpha_{2,e}(tm + n),$$

which completes the induction step and the proof of the theorem.  $\square$

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